Section S9 FINANCIAL PHYSICS

1. Discuss the extent to which financial market data show scaling properties. [10]

Describe the microstructure of financial markets, focusing on the order of events which lead to the determination of the price. [8]

~~Explain how multi-agent models can be built to reproduce the stylized facts observed in financial market data.~~ [7]

2. Describe and compare forward contracts, European options and American options. [8]

Derive the Black-Scholes equation for the price of a European call option, explaining carefully any approximations that you make. Describe the resulting hedging strategy. Discuss how the hedging strategy and the value of the call option vary as the option approaches expiry. [17]

3. Give an account of the concept of risk in financial markets. [6]

~~Discuss the extent to which it pays to be in the minority or majority in a financial market.~~ [5]

The change Δx_n in the price of a particular stock on day n is described by the following equation:

$$\Delta x_n = \delta_n + \delta_{n-1}\delta_{n-2}$$

where the random variable δ_k takes values -1 or $+1$ with equal probability. Evaluate (a) $\langle \Delta x_n \rangle$, (b) $\langle \Delta x_n \Delta x_{n-1} \Delta x_{n-2} \rangle$, and (c) $\langle \Delta x_{n'} \Delta x_n \rangle$ for $n' \neq n$. Discuss the duration and magnitude of the largest possible crash that can occur for this stock. Compare these results, together with those from cases (a), (b) and (c), with the corresponding ones for a random-walk model of stock-price movements. [14]

~~**4.** The basic Minority Game describes N financial market traders who each repeatedly decide whether to buy or sell. The previous m winning decisions are used by each trader in forming his next decision. Write down the full strategy space for the cases $m = 1$ and $m = 2$. In each case, indicate a set of strategies that form a reduced strategy space. Sketch the directed graph representing the dynamics of the m-bit history string for $m = 1$, $m = 2$ and $m = 3$. Explain how the dynamics on these directed graphs could lead to a significant price movement.~~ ~~[9]~~

~~The excess demand $D(t)$ is given by the difference between the numbers of buyers and sellers at time t. Assuming that all traders have two strategies, which are chosen randomly from the reduced strategy space, discuss the variation in the standard deviation σ of $D(t)$ as the bit-string length m is increased.~~ [5]

~~For small m, derive an approximate formula describing σ as a function of m. Deduce an approximate expression for the value of m beyond which σ falls below the value that it would have for a population of coin-tossing traders. Evaluate your expression for the cases $N = 101$ and $N = 201$.~~ ~~[11]~~

$$\left[\ \sum_{r=1}^{n} r^2 = \frac{1}{6}\cancel{n(n+1)(2n+1)} \qquad \sum_{r=1}^{n} r = \frac{1}{2}\cancel{n(n+1)}\ \right]$$

Section S9 FINANCIAL PHYSICS

1. Discuss the microstructure of financial markets, focusing on the order of events that lead to the determination of a price. Describe the extent to which multi-agent models can account for the stylized facts observed in financial market price data. [12]

Under certain conditions, the volatility of price increments over n timesteps, σ_n, can be related to the volatility over a single timestep σ, by the expression $\sigma_n = n^{1/2}\sigma$. Give the derivation of this expression, stating clearly the conditions under which it will hold true. [7]

In a particular market, the price data are found to satisfy the generalised relationship $\sigma_n = n^{\alpha}\sigma$ with $\alpha > 1/2$. Discuss the implications of this finding. [6]

2. ~~Compare the dynamics that could arise in economic models represented by~~

~~(a) one-dimensional,~~

~~(b) two-dimensional, and~~

~~(c) three-dimensional~~

~~non-linear continuous-time equations. How would your answers change if the equations were discrete in time?~~ ~~[10]~~

~~The prices at time t of two particular stocks A and B, are given by the positive quantities $a(t)$ and $b(t)$. These prices evolve in time according to the following equations:~~

$$2\dot{a} = a(F - a - 2b)\,,$$

$$2\dot{b} = b(3 - a - b)\,,$$

~~where F is a parameter. For the cases $F = 4$~~ and $F = 8$, find the fixed points and ~~sketch the phase portrait in the (a, b) plane.~~ ~~[15]~~

3. A trader in a given stock market X has a probability of winning at time t determined by his success in any market at the previous two timesteps $t-2$ and $t-1$. When he wins (W) he earns one unit of cash. When he loses (L) he pays one unit of cash. Following a sequence of outcomes (L, L) at timesteps $(t-2,\, t-1)$, his probability of winning at timestep t is p_{LL}. Following (L, W) it is p_{LW}, following (W, L) it is p_{WL} and following (W, W) it is p_{WW}. Deduce his probability of winning for a generic run of trades in the stationary regime. Derive a condition involving p_{LL}, p_{LW}, p_{WL} and p_{WW}, such that he loses on average. [9]

Although he is losing on average in market X, he decides to invest in a futures market Y where his probability of winning in any timestep is a fixed number p which is less than 1/2. He switches randomly between the two markets X and Y. Derive the condition under which he can now win on average. Give an explanation of this phenomenon in terms of the system's dynamics. [8]

Consider $p = \frac{1}{2} - \alpha$, $p_{LL} = \frac{9}{10} - \alpha$, $p_{LW} = p_{WL} = \frac{1}{4} - \alpha$ and $p_{WW} = \frac{7}{10} - \alpha$ where $\alpha \geq 0$. Derive the range of values of α such that, by switching randomly between the two losing markets X and Y, he wins on average. [8]

4. Explain the terms *implied volatility*, *American option* and *European option*. [6]

Sketch the pay-off diagrams at expiry for a European call option and a European put option, as a function of the asset price. Compare these with the pay-off diagrams for the long and the short positions in a forward contract. [6]

Derive an expression for the price of a European call option that is valid for non-Gaussian markets. Give an outline derivation of the hedging strategy which will minimize the risk. State clearly any assumptions you make. [13]

Section S9 FINANCIAL PHYSICS

1. Discuss what is meant by the financial term *volatility*. [3]

Sketch the pay-off diagrams at expiry for long and short positions in a forward contract as a function of the asset price. Compare these with the pay-off diagrams for a European call and put option. [5]

~~Consider the basic Minority Game as a model of financial markets. At each timestep, N traders observe the previous m price movements in order to decide whether to buy (action $+1$) or sell (action -1). Upward and downward price movements are represented by 1 and 0 respectively. Write out the full strategy space for the cases $m = 1$ and $m = 2$. For $m = 2$ indicate a subset of strategies that form a reduced strategy space and comment on the correlations between these strategies in terms of their relative Hamming distance. Sketch the De Bruijn graph representing the dynamics of the m-bit history string for the cases $m = 1$, $m = 2$ and $m = 3$. For the case $m = 2$ discuss the dynamical cycles that might arise. [10]~~

~~Consider the time-series $n_{+1}(t)$ corresponding to the number of traders choosing to buy at time t. Each trader has access to the previous m outcomes and has two strategies chosen randomly from the reduced strategy space. Discuss the variation in the standard deviation σ of $n_{+1}(t)$ as the bit-string length m is increased~~. [7]

2. In the context of financial markets, explain what is meant by the terms *random walk* and *stylized facts*. [8]

The monthly increment of a particular market index can be described by the following equation:

$$\delta y(t) = \alpha(t-2)\alpha(t-1) + \alpha(t),$$

where at each timestep t the random variable α takes values -1 or $+1$ with equal probability. Evaluate the following expectation values for the process $\delta y(t)$:

(a) $\mathrm{E}[\delta y(t)]$,

(b) $\mathrm{E}[\delta y(t')\delta y(t)]$ for $t' \neq t$,

(c) $\mathrm{E}[\delta y(t)\delta y(t-1)\delta y(t-2)]$. [10]

Comment on the predictability of the time-series $y(t)$. Derive the magnitude and duration of the worst possible drawdown that can occur for this index. Compare these results with the worst possible drawdown for another index, the monthly increment of which is given by $\delta y(t) = \alpha(t)$. [7]

3. Give an account of **each** of the following topics:

(a) Scaling of financial market data. [8]

(b) Black–Scholes option pricing theory. [8]

(c) Option pricing in non-Gaussian markets. [9]

Section K. PHYSICS AND FINANCE

K1. ~~Give a brief account of the types of dynamics that can arise in one, two and three-dimensional non-linear systems described in continuous time. Comment on the differences which would arise with a discrete time description.~~ ~~[8]~~

~~The exchange rates of currencies X and Y with the US dollar, are given by the positive quantities~~ $x(t)$ ~~and~~ $y(t)$ ~~at time~~ t. ~~Their co-evolution is described by the following time-dependent equations~~

$$\begin{aligned} 2\dot{x} &= x(M - x - 2y) \\ 2\dot{y} &= y(4 - x - y). \end{aligned}$$

~~The parameter~~ M ~~is determined by US bond rates. For each of the cases (a)~~ $M = 6$ ~~and (b)~~ $M = 3$, ~~find the fixed points, their stabilities, and sketch the phase portrait in the~~ (x, y) ~~plane.~~ ~~[14]~~

~~Compare the temporal evolution of the two exchange rates for cases (a) and (b).~~ ~~[3]~~

K2. Explain the assumptions made in the derivation of the Black-Scholes equation for pricing options. Discuss the form of the Black-Scholes hedging strategy for a European call option as a function of asset price, during the option's lifetime. [9]

By considering the variation in wealth of the option writer, derive an expression for the price of a European call option in non-Gaussian markets. State clearly any assumptions you make. Give an outline derivation of the hedging strategy which minimizes the risk. [16]

K3. Give an account of the microstructure of financial markets, focusing on the order of events which lead to the determination of the price. [10]

~~Discuss the extent to which multi-agent models can account for the stylized facts observed in financial market data.~~ [9]

Explain the idea of scaling as applied to financial market data. [6]

K4. Discuss the implications of the *independent and identically distributed* (i.i.d.) assumption invoked by standard finance theory when describing price-increments in a market. [5]

Under certain conditions the volatility σ_n of price-increments over n timesteps can be related to the volatility σ over a single timestep by the expression $\sigma_n = n^{1/2}\sigma$. Give the derivation of this expression, stating clearly the conditions under which it will hold true. [6]

It is often found that empirical market data satisfy the generalized relationship $\sigma_n = n^{\alpha}\sigma$ with $\alpha \neq 1/2$. Discuss the implications of this finding. [4]

Show that under the assumptions of standard finance theory, the probability distribution of prices satisfies a diffusion equation. Comment on the general form of the solution to this equation, and discuss how it is related to the distribution of price-increments for a single timestep in a simple coin-toss market. [10]

K. PHYSICS AND FINANCE

K1. Discuss briefly the concepts of *volatility* and *risk*. Explain the terms *implied volatility*, *American option*, *European option* and *lookback option*. [6]

Sketch the pay-off diagrams at expiry for a European call and put option, as a function of the asset price. How do these differ from the pay-off diagrams for long and short positions in a forward contract? [5]

Using Ito's lemma, or otherwise, derive the Black-Scholes partial differential equation for the price of a European call option. Discuss any approximations made. Explain the resulting hedging strategy. Sketch the value of the call option both before and at expiry, as a function of the exercise price. [12]

It has been claimed that derivatives trading is a major cause of financial crashes. Comment on this statement. [2]

K2. What does the efficient market hypothesis imply about temporal correlations in financial price series? Give an outline derivation of a pricing theory for European call options in a non-Gaussian market. [8]

The daily price increment of a particular dot-com stock can be described by the following equation:

$$\delta p(t) = \epsilon(t) + \epsilon(t-1)\epsilon(t-2)$$

where at each timestep t, the random variable ϵ takes values $+1$ or -1 with equal probability. Verify that the expectation values for the process $\delta p(t)$ satisfy $\mathrm{E}[\delta p(t)] = 0$ and $\mathrm{E}[\delta p(t)\delta p(t')] = 0$ for $t \neq t'$. [4]

Show that the three-point correlation function $\mathrm{E}[\delta p(t-2)\delta p(t-1)\delta p(t)]$ is non-zero, and evaluate the conditional expectation value $\mathrm{E}[\delta p(t)|\delta p(t-2), \delta p(t-1)]$. Comment on the predictability of the price series $p(t)$. [7]

Derive the magnitude and duration of the worst possible drawdown that can occur for this stock. Compare these results to the worst possible drawdown for a blue-chip stock whose daily price increment is given by $\delta p(t) = \epsilon(t)$. [6]

K3. Explain the concept of *stylized facts* for financial markets. Discuss how you would determine whether a given price process $p(t)$ constitutes a random walk. [6]

An investor trading in a given currency market A, has a probability of winning at time t determined by her success at the previous two timesteps $t-2$ and $t-1$. When she wins (W) she earns one unit of cash. When she loses (L) she pays one unit of cash. Following a sequence of outcomes (L,L) at timesteps ($t-2$,$t-1$), her probability of winning at timestep t is p_a. Following (L,W) it is p_b. Following (W,L) it is p_c. Following (W,W) it is p_d. Obtain her probability to win for a generic run of trades in the stationary regime. Derive a condition involving p_a, p_b, p_c and p_d, such that she loses on average. [7]

Although she is losing on average in market A, she decides to invest in a commodity market B where her probability of winning $p < 1/2$. She switches randomly between the two markets A and B. Derive the condition under which she can now win on average. Give an explanation of this phenomenon in terms of the system's dynamics. [7]

Consider $p = 1/2 - \delta$, $p_a = 9/10 - \delta$, $p_b = p_c = 1/4 - \delta$ and $p_d = 7/10 - \delta$ where $\delta \geq 0$. Derive the range of values of δ such that, by switching randomly between the two losing markets A and B, she wins on average. [5]

~~**K4.** By considering a round-trip of trades, show that it can pay to be in the minority in a financial market. [4]~~

~~Consider a simple, minority model of a financial market where N traders have to decide at each timestep whether to buy (choice 1) or sell (choice 0). Each trader observes the previous m winning outcomes in order to make her decision. Write out the full strategy space for the case $m = 2$. Indicate a subset of strategies that form a reduced strategy space, and comment on the correlations between these strategies in terms of their relative Hamming distance. For the cases $m = 1$, $m = 2$ and $m = 3$, sketch the directed graph representing the dynamics of the m-bit history string. For the case $m = 3$, explain the dynamics on this directed graph that could lead to a significant drawdown or crash. [9]~~

~~Consider the time-series $N_1(t)$ produced by the number of traders choosing to buy at time t. Each trader has access to the previous m outcomes, and has two strategies chosen randomly from the strategy space. Describe and explain the variation in the standard deviation σ of $N_1(t)$ as the bit-string length m is increased.~~ [5]

~~For small m, derive an approximate formula describing σ as a function of m. Hence obtain an approximate expression for the value of m beyond which σ falls below σ_{rand}, where σ_{rand} is the standard deviation in a population of N coin-tossing traders. Evaluate this expression for the case $N = 101$. [7]~~

$$\left[\ \sum_{r=1}^{n} r = \frac{1}{2}n(n+1) \quad \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)\ \right]$$